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Derivation of a model for symmetrized electromagnetism from a space–time with torsion

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Abstract. The most general covariant spinor derivative compatible with the Minkowski line element leads in a unique way to an antisymmetric torsion tensor. It turns out that this can be expressed by means of a pseudovector potential, counterpart of the usual vector potential of electrodynamics. In this way some of the models for a symmetrized theory of the electromagnetic interaction with spin-$\frac{1}{2}$ particles are rederived.

1. Introduction

In classical as well as in quantum mechanics the electromagnetic forces are introduced by means of the vector potential A_α through the relation

$$mv_\alpha = p_\alpha - eA_\alpha, \tag{1.1}$$

where v_α, m and e represent respectively the velocity, the mass and the charge of the particle. p_α and q_β satisfy the relation

$$\{p_\alpha, q_\beta\} = -\delta_{\alpha\beta}, \tag{1.2}$$

where the bracket denotes the Poisson bracket in the classical case and stands for commutator multiplied by $1/i\hbar$ in the quantum case. In both theories we have

$$\{mv_\alpha, mv_\beta\} = -eF_{\alpha\beta}, \tag{1.3}$$

where $F_{\alpha\beta} = A_{\alpha,\beta} - A_{\beta,\alpha}$ denotes the electromagnetic field. Looking for a geometrical description of the electromagnetic forces, the essential point of the above equation is the non-commutativity of the velocity operators. This can be compared with the non-commutativity of the covariant derivatives which is responsible for the existence of the gravitational field in Riemannian space–time. We recall that a general metric space is described by a metric tensor g_{ij} and a linear connection Γ^k_{ij} which are in the first instance independent. The geometrical content of space–time is then completely characterized by the metric tensor and by the tensors

$$\underset{\vee}{\Gamma^k_{ij}} = \tfrac{1}{2}(\Gamma^k_{ij} - \Gamma^k_{ji}), \tag{1.4}$$

representing the torsion and

$$R^{\nu}_{\mu\sigma\tau} = \partial_\tau \Gamma^\nu_{\mu\sigma} - \partial_\sigma \Gamma^\nu_{\mu\tau} + \Gamma^\lambda_{\mu\sigma}\Gamma^\nu_{\lambda\tau} - \Gamma^\lambda_{\mu\tau}\Gamma^\nu_{\lambda\sigma}, \tag{1.5}$$

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representing the curvature. As the sources of the electromagnetic field are leptons and other elementary particles, a geometrical theory of the electromagnetic interaction should be written down first for particles with spin-$\frac{1}{2}$. In this article we restrict ourselves to this subject. The essential content of equation (1.3) must appear on account of covariant differentiations operating on spin-$\frac{1}{2}$ spinors.

In conventional theories of electromagnetism and gravitation the electromagnetic field is described as an external field in a Riemannian space–time. In order to remain as close as possible to the existing theories of electrodynamics and gravitation we reserve the dependence of the metric tensor entirely for the description of gravitation. Neglecting the gravitational interactions, the line element reduces to that of Minkowski space. Hence, we seek for the most general covariant differentiations (acting on spinors), compatible with the Minkowski line element. The torsionless case is reviewed by Bade and Jehle (1953). The general analysis admits an antisymmetrical torsion tensor. It turns out that this torsion tensor is related to an axial vector potential M_μ appearing in a symmetrical theory of the electromagnetic interaction, while the usual vector potential A_μ can still be interpreted as a gauge field.

2. Non-symmetrical connections compatible with Minkowski geometry

A linear affine connection of the space–time can be most generally written as

$$\Gamma^k_{ij} = \left\{ {k \atop ij} \right\} + U^k_{ij}, \tag{2.1}$$

where $\{ {k \atop ij} \}$ is Christoffel's symbol of the second kind belonging to the metric $g_{\alpha\beta}$, and the U^k_{ij} are functions of the coordinates which are so far unspecified and which may contain a symmetrical as well as an antisymmetrical part with respect to the lower indices. The expression for the covariant derivative of the metrical tensor $g_{\alpha\beta}$ with respect to the affine connection (2.1) is given by:

$$D_\rho g_{\mu\nu} = g_{\mu\nu,\rho} - \left\{ {\sigma \atop \mu\rho} \right\} g_{\sigma\nu} - \left\{ {\sigma \atop \nu\rho} \right\} g_{\mu\sigma} - U^\sigma_{\mu\rho} g_{\sigma\nu} - U^\sigma_{\nu\rho} g_{\mu\sigma}, \tag{2.2}$$

where the comma denotes partial differentiation with respect to the space–time coordinates. In the case where the metrical tensor $g_{\alpha\beta}$ is equal to the Minkowski tensor $\eta_{\alpha\beta}$, the equation (2.2) reduces to the expression

$$D_\rho \eta_{\mu\nu} = -U_{\mu\rho|\nu} - U_{\nu\rho|\mu}, \tag{2.3}$$

where the following notation has been introduced:

$$U_{\mu\rho|\nu} = U^\sigma_{\mu\rho}\eta_{\sigma\nu}. \tag{2.4}$$

For reasons of simplicity the following metric has been chosen:

$$\eta_{\alpha\beta} = \mathrm{diag}(1, -1, -1, -1).$$

By the gauge conservation condition

$$D_\rho \eta_{\mu\nu} = 0, \tag{2.5}$$

and by taking into account equation (2.3) we find the following restriction on $U_{\mu\nu|\rho}$:

$$U_{\mu\rho|\nu} = -U_{\nu\rho|\mu}. \tag{2.6}$$

The covariant derivative of a spinor ψ is defined as

$$\vec{D}_\rho \psi = \psi_{,\rho} - \Gamma_\rho \psi \equiv (\vec{\partial}_\rho - \Gamma_\rho)\psi, \tag{2.7}$$

while

$$\bar{\psi}\overleftarrow{D}_\rho = \bar{\psi}_{,\rho} + \bar{\psi}\Gamma_\rho \equiv \bar{\psi}(\overleftarrow{\partial}_\rho + \Gamma_\rho), \tag{2.8}$$

where Γ_ρ is the so called spinor connection. All rules concerning ordinary derivatives remain valid for covariant derivatives, if the covariant differentiation operators occurring everywhere are adjusted to the nature of the quantity on which they act. Taking this into account and putting further the covariant derivatives of the Dirac γ_α and γ^α spinor operators equal to zero, one can write (Tonnelat 1965):

$$D_\rho \gamma_\mu \equiv \gamma_{\mu,\rho} - U^\sigma_{\mu\rho}\gamma_\sigma - (\Gamma_\rho \gamma_\mu - \gamma_\mu \Gamma_\rho) = 0, \tag{2.9}$$

$$D_\rho \gamma^\mu \equiv \gamma^\mu{}_{,\rho} - U^\mu_{\sigma\rho}\gamma^\sigma - (\Gamma_\rho \gamma^\mu - \gamma^\mu \Gamma_\rho) = 0. \tag{2.10}$$

Using a representation in which the Dirac matrices are constant, we have:

$$\gamma_{\mu,\rho} = \gamma^\mu{}_{,\rho} = 0. \tag{2.11}$$

Equations (2.9), (2.10) and (2.11) yield the definition for the spinor connection Γ_ρ. The compatibility requirement for equations (2.9)–(2.11) arises in a supplementary condition for $U_{\mu\nu|\rho}$

$$U_{\mu\nu|\rho} = -U_{\nu\mu|\rho}. \tag{2.12}$$

The equations (2.6) and (2.12) imply that $U_{\mu\nu|\rho}$ has to be a tensor antisymmetrical in all its indices. This allows one to write:

$$U_{\mu\nu|\rho} = \epsilon_{\mu\nu\rho\lambda}L^\lambda, \tag{2.13}$$

with L^λ an arbitrary pseudovector and $\epsilon_{\mu\nu\rho\lambda}$ the complete real antisymmetrical tensor. As another consequence of (2.12) it is found that:

$$\underset{\vee}{\Gamma^\sigma_{\mu\nu}} = U^\sigma_{\mu\nu} = \Gamma^\sigma_{\mu\nu}. \tag{2.14}$$

So it is clear that $U^\sigma_{\mu\nu}$ is completely defined by the torsion of the space. Furthermore, it is easily verified that the spinor connection calculated from equation (2.9)—or equivalently from equation (2.10)—turns out to be:

$$\Gamma_\rho = C_\rho I + \tfrac{1}{4}\mathrm{i}\sigma^{\mu\nu}U_{\mu\nu|\rho} = C_\rho I + \tfrac{1}{2}\gamma_5\sigma_{\rho\lambda}L^\lambda, \tag{2.15}$$

where C_ρ is an arbitrary four-vector and I is the identity operator in spinor space. Finally we note that the Riemannian curvature tensor expressed in terms of the pseudovector L^λ becomes:

$$R^\nu_{\mu\sigma\tau} = \epsilon_{\mu\sigma}{}^\nu{}_\lambda L^\lambda{}_{,\tau} - \epsilon_{\mu\tau}{}^\nu{}_\lambda L^\lambda{}_{,\sigma} + (\epsilon_{\mu\sigma}{}^\lambda{}_\rho \epsilon_{\lambda\tau}{}^\nu{}_\eta - \epsilon_{\mu\tau}{}^\lambda{}_\rho \epsilon_{\lambda\sigma}{}^\nu{}_\eta)L^\rho L^\eta, \tag{2.16}$$

which is clearly different from zero. Contraction gives

$$R_{\mu\tau} = R^\sigma_{\mu\sigma\tau} = -\epsilon_{\mu\tau}{}^\sigma{}_\lambda L^\lambda{}_{,\sigma} - 2\eta_{\mu\tau}L^2 + 2L_\mu L_\tau, \tag{2.17}$$

and finally

$$R = -6L^2. \tag{2.18}$$

3. The Lagrangian density for spin-$\frac{1}{2}$ particles

For the special relativistic free Lagrangian density of a massive free spin-$\frac{1}{2}$ particle, the usual expression reads:

$$\mathscr{L}_0 = \tfrac{1}{2}\mathrm{i}\bar{\psi}\gamma^\mu \overleftrightarrow{\partial}_\mu \psi - m\bar{\psi}\psi, \tag{3.1}$$

where the symbol $\overleftrightarrow{\partial}_\mu$ stands for:

$$A\overleftrightarrow{\partial}_\mu B = A(\vec{\partial}_\mu B) - (A\overleftarrow{\partial}_\mu)B. \tag{3.2}$$

Instead of taking the partial derivatives in equation (3.1), we substitute the covariant ones of equations (2.7) and (2.8) in the sense of minimal coupling to the torsion. This results in:

$$\mathscr{L} = \tfrac{1}{2}\mathrm{i}\bar{\psi}\gamma^\mu \overleftrightarrow{D}_\mu \psi - m\bar{\psi}\psi = \mathscr{L}_0 - \mathrm{i}\bar{\psi}\gamma^\mu \Gamma_\mu \psi, \tag{3.3}$$

where the last term can be considered as an interaction Lagrangian density. By taking into account equation (2.15) and setting:

$$C_\rho = -\mathrm{i}eA_\rho, \tag{3.4}$$

$$L^\lambda = \tfrac{2}{3}qM^\lambda, \tag{3.5}$$

where e and q are real constants so far undetermined, the Lagrangian density $\mathscr{L}_{\rm int}$ can be written down in terms of the vectors A_μ and M_μ:

$$\mathscr{L}_{\rm int} = -\bar{\psi}\gamma^\mu (eA_\mu + \tfrac{1}{3}\mathrm{i}q\gamma_5\sigma_{\mu\lambda}M^\lambda)\psi,$$

or

$$\mathscr{L}_{\rm int} = -eA_\mu\bar{\psi}\gamma^\mu\psi + qM_\mu\bar{\psi}\gamma^\mu\gamma_5\psi. \tag{3.6}$$

In order to have a Hermitian Lagrangian density, it is necessary that A_μ and M_μ are real four-vectors. To see this, one immediately verifies that

$$(-e\bar{\psi}\gamma^\mu A_\mu\psi + q\bar{\psi}\gamma^\mu\gamma_5 M_\mu\psi)^\dagger = -e\bar{\psi}\gamma^\mu A_\mu^*\psi + q\bar{\psi}\gamma^\mu\gamma_5 M_\mu^*\psi,$$

and comparison with (3.6) proves the statement.

The field equation derived from $\mathscr{L}_0 + \mathscr{L}_{\rm int}$ reads

$$\mathrm{i}\gamma^\mu \vec{D}_\mu\psi = m\psi, \tag{3.7}$$

or

$$\mathrm{i}\gamma^\mu\psi_{,\mu} - \gamma^\mu(eA_\mu - q\gamma_5 M_\mu)\psi = m\psi.$$

Looking at any of the equivalent expressions (3.6) for the Lagrangian density $\mathscr{L}_{\rm int}$, it is striking that apart from the M_μ dependent term, the Lagrangian density is formally the same as the one describing the motion of a quantized spin-$\frac{1}{2}$ particle with mass m and charge e in a classical electromagnetic field with four-potential A_μ.

A second four-vector namely M_μ, is coupled to an axial four-vector and is, in contrast to A_μ related to the torsion of the space directly. Such a four-vector also appears in the symmetrized formulation of electromagnetism. In this widely discussed theory a so called m-electric four-potential is introduced (Cabibbo and Ferrari 1962, Leiter 1970).

We introduce the notation

$$f_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu}, \qquad h_{\mu\nu} = M_{\mu,\nu} - M_{\nu,\mu}. \tag{3.8}$$

The electromagnetic tensor $F_{\mu\nu}$ in the symmetrized formulation of electromagnetism is then defined by

$$F_{\mu\nu}=f_{\mu\nu}+\tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta}h^{\alpha\beta}, \tag{3.9}$$

whereas the dual electromagnetic tensor $\tilde{F}_{\mu\nu}$ is found to be

$$\tilde{F}_{\mu\nu}=-h_{\mu\nu}+\tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta}f^{\alpha\beta}. \tag{3.10}$$

The Maxwell equations can be written in the form

$$\partial^{\nu}F_{\mu\nu}=J_{\mu}, \tag{3.11}$$

$$\partial^{\nu}\tilde{F}_{\mu\nu}=\tilde{J}_{\mu}, \tag{3.12}$$

where J_μ and $\tilde{J}_\mu$ are respectively an electromagnetic current and a pseudovector magnetic four-current. The basic equations (3.11) and (3.12) are the fundamental ingredients of a monopole theory if the electrical current J_μ and the magnetic current $\tilde{J}_\mu$, which provides a monopole source for the electromagnetic field, are both conserved. As in that case the Lorentz gauge condition can be implied on A_μ and M_μ, the Maxwell equations (3.11) and (3.12) reduce to

$$\Box A_{\mu}\equiv\partial^{\nu}\partial_{\nu}A_{\mu}=J_{\mu}, \tag{3.13}$$

$$\Box M_{\mu}\equiv\partial^{\nu}\partial_{\nu}M_{\mu}=-\tilde{J}_{\mu}. \tag{3.14}$$

These equations are the Euler–Lagrange equations corresponding to the variational principle with the Lagrangian density:

$$\mathscr{L}_{\mathrm{EM}}=-\tfrac{1}{8}F_{\mu\nu}F^{\mu\nu}+\tfrac{1}{8}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}-A_{\mu}J^{\mu}-M_{\mu}\tilde{J}^{\mu}. \tag{3.15}$$

The last two terms in the right-hand side of (3.15) have their equivalent in the Lagrangian density (3.6).

Identifying the current-dependent parts, it follows immediately that J_μ and $\tilde{J}_\mu$ are given by

$$J_{\mu}=e\bar{\psi}\gamma_{\mu}\psi, \tag{3.16}$$

$$\tilde{J}_{\mu}=-q\bar{\psi}\gamma_{\mu}\gamma_{5}\psi, \tag{3.17}$$

where the real constant q, so far arbitrary, becomes a constant with the dimension of a charge.

From the equation (3.7) and its Hermitian conjugate we derive that

$$J^{\mu}{}_{,\mu}=0, \tag{3.18}$$

$$\tilde{J}^{\mu}{}_{,\mu}=-2\mathrm{i}mq\bar{\psi}\gamma_{5}\psi. \tag{3.19}$$

Hence $\tilde{J}_\mu$ is not conserved unless m or q equals zero. For $m\neq 0$, q necessarily vanishes, and the particle (described by the field ψ_{I}) can only carry electric charge. For the description of magnetic charge ($q\neq 0$) we must introduce a second field ψ_{II} for which the bare mass is zero. Following Schwinger (1966) and Hagen (1965) we suppose that this second field does not carry electric charge. The resulting Lagrangian density is now

$$\begin{aligned}\mathscr{L}=&-\tfrac{1}{8}F_{\mu\nu}F^{\mu\nu}+\tfrac{1}{8}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}+\tfrac{1}{2}\mathrm{i}\bar{\psi}_{\mathrm{I}}\gamma^{\mu}\overleftrightarrow{\partial}_{\mu}\psi_{\mathrm{I}}-eA^{\mu}\bar{\psi}_{\mathrm{I}}\gamma_{\mu}\psi_{\mathrm{I}}-m\bar{\psi}_{\mathrm{I}}\psi_{\mathrm{I}}\\&+\tfrac{1}{2}\mathrm{i}\bar{\psi}_{\mathrm{II}}\gamma^{\mu}\overleftrightarrow{\partial}_{\mu}\psi_{\mathrm{II}}+qM^{\mu}\bar{\psi}_{\mathrm{II}}\gamma_{\mu}\gamma_{5}\psi_{\mathrm{II}}.\end{aligned} \tag{3.20}$$

If one wishes to avoid the doubling of the usual number of degrees of freedom of the electromagnetic field, one can express M^μ by means of the source terms in the free part of the Lagrangian. The appropriate Lagrangian is then essentially the linear model of Boulware and Gilbert (1962). The invariance of the Lagrangian (3.20) under the local gauge transformation

$$\psi_{\mathrm{I}} \to \mathrm{e}^{+\mathrm{i}\lambda_{\mathrm{I}}}\psi_{\mathrm{I}}, \qquad \bar{\psi}_{\mathrm{I}} \to \bar{\psi}_{\mathrm{I}}\,\mathrm{e}^{-\mathrm{i}\lambda_{\mathrm{I}}}, \tag{3.21}$$

$$A_\mu \to A_\mu - e^{-1}\partial_\mu \lambda_{\mathrm{I}}, \tag{3.22}$$

results in the charge conservation law (3.18).

The Lagrangian is also invariant under the group of gauge transformations

$$\psi_{\mathrm{II}} \to \mathrm{e}^{+\mathrm{i}\lambda_{\mathrm{II}}\gamma_5}\psi_{\mathrm{II}}, \qquad \bar{\psi}_{\mathrm{II}} \to \bar{\psi}_{\mathrm{II}}\,\mathrm{e}^{+\mathrm{i}\lambda_{\mathrm{II}}\gamma_5}, \tag{3.23}$$

$$M_\mu \to M_\mu + q^{-1}\partial_\mu \lambda_{\mathrm{II}}. \tag{3.24}$$

Following Noether's theorem, the latter invariance gives rise to the conservation law

$$\tilde{J}^{\mu}{}_{,\mu} = 0. \tag{3.25}$$

The introduction of a mass term of the form $m_{\mathrm{II}}\bar{\psi}_{\mathrm{II}}\psi_{\mathrm{II}}$ in (3.20) would have destroyed the invariance of the Lagrangian on account of the transformation (3.23).

The field equation for ψ_{I} is the usual Dirac equation with conservation law

$$\partial_\mu(\bar{\psi}_{\mathrm{I}} P_{\mathrm{I}}^{\mu} \psi_{\mathrm{I}}) = 0, \tag{3.26}$$

where $P_\mu^{\mathrm{I}} = \mathrm{i}\partial_\mu - eA_\mu$. The analogous law for the field ψ_{II} is

$$\partial_\mu(\bar{\psi}_{\mathrm{II}} P_{\mathrm{II}}^{\mu} \psi_{\mathrm{II}}) = 0, \tag{3.27}$$

where P_μ^{II} is now given by

$$P_\mu^{\mathrm{II}} = \mathrm{i}\partial_\mu - \mathrm{i}q\gamma_5\sigma_{\mu\lambda}M^{\lambda}. \tag{3.28}$$

The q-dependent part of P_μ^{II} differs from Γ_μ (with $e=0$) upon a factor of $\frac{1}{3}$.

The analogue to the mean equation of motion

$$\partial_\mu(\bar{\psi}_{\mathrm{I}}\gamma^\mu P_\beta^{\mathrm{I}}\psi_{\mathrm{I}}) = J^\alpha f_{\alpha\beta}, \tag{3.29}$$

is

$$\partial_\mu(\bar{\psi}_{\mathrm{II}}\gamma^\mu P_\beta'^{\mathrm{II}}\psi_{\mathrm{II}}) = \tilde{J}^\alpha h_{\alpha\beta}, \tag{3.30}$$

with

$$P_\beta'^{\mathrm{II}} = \mathrm{i}\partial_\beta + q\gamma_5 M_\beta, \tag{3.31}$$

different from P_β^{II}. The Lorentz forces on the right-hand sides of equations (3.29) and (3.30) are completely analogous to the expression for the Lorentz force derived in the case of the motion of a classical electrically and magnetically charged particle in a symmetrized electromagnetic field.

4. Conclusion

In the literature several models for a symmetrized form of electromagnetism are proposed. Some of these models can be rederived by introducing a general covariant spinor derivative compatible with the Minkowski line element. Indeed, this requirement leads in a unique way to an antisymmetric torsion tensor, which can be expressed by means of a pseudovector potential, counterpart of the usual vector potential in a symmetrized version of electrodynamics. The conservation law for the magnetic current implies that the bare mass of a particle carrying magnetic current must vanish. A particle with non-vanishing bare mass can carry electric charge only. In this way the essence of the linear model of Boulware and Gilbert is rederived from geometrical considerations.

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